$E \equiv x - 2 = -y = \frac{z+3}{3}$

Or $E \equiv \begin{cases} x + y = 2 \\ 3x + z - 2 = 0 \end{cases}$

c) If θ is the angle between P and E then

$\sin\theta = \frac{\overline{nu}}{|\overline{u}|.|\overline{v}|} = 1$

since $\theta = \frac{\pi}{2}$

The result surprised that if D//E and D⊥P then E⊥P

d) $\int \frac{x^3}{\sqrt{4+x^2}}dx$

Let $t^2 = 4+x^2 \Rightarrow 2tdt = 2xdx$

$\int \frac{x^3}{\sqrt{4+x^2}}dx = \int \frac{x^2}{\sqrt{4+x^2}}xdx = \int \frac{(t^2-4)}{t}tdt = \int (t^2-4)dt = \frac{t^3}{3} - 4t + c$

$= \frac{4+x^2}{3}\sqrt{x^2+4} - 4\sqrt{x^2+4} + C = \sqrt{x^2+4}(\frac{4+x^2}{3} - 4) + C$

ADVENCED LEVEL MATHEMATICS NATIONAL EXAMINATION PAPER 2007

SECTION A: Attempt all questions. (55 marks)

01. The tangent to the graph of function $f(x) = \frac{x^2+mx-3}{(m-1)x+1}$ at x=0 is parallel to the line y=5x-4. Find the value of m. 3 marks

Answer:

$f(x) = \frac{x^2+mx-3}{(m-1)x+1}$

$f'(x) = \frac{(2x+m)[(m-1)x+1]-(m-1)[x^2+mx-3]}{[(m-1)x+1]^2}$

The angular coefficient of T at point x = 0 is equal to

$f'(0) = \frac{(0+m)[(m-1)0+1]-(m-1)[0+0-3]}{[(m-1)x+1]^2} = \frac{m+3m-3}{1^2} = 4m-3$

The same as T is parallel to $D \equiv y = 5x-4$

02. Find the equation of the tangent to the curve given by the parametric equations $\begin{cases} x = a\cos^3 t \\ y = a\sin^3 t \end{cases}$ at point $t = \frac{\pi}{4}$. 3 marks

Answer:

$X = a\cos^3 t$

$y = a\sin^3 t$

$y' = \frac{dy}{dx} = \frac{3a\sin^2 t\cos t\, dt}{-3a\cos^2 t\sin t\, dt} = -\frac{\sin t}{\cos t} = -\tan t$

$T \equiv y = y_o + y'_0(x - x_0)$

$y_0 = y\left(\frac{\pi}{4}\right) = a\sin^3\left(\frac{\pi}{4}\right) = a\left(\frac{\sqrt{2}}{2}\right)^3 = a\frac{\sqrt{2}}{4}$

$y'_0 = y'\left(\frac{\pi}{4}\right) = -\tan\frac{\pi}{4} = -1$

$x_0 = a\cos^3\left(\frac{\pi}{4}\right) = a\frac{\sqrt{2}}{4}$

$T \equiv y = a\frac{\sqrt{2}}{4} - 1\left(x - a\frac{\sqrt{2}}{4}\right)$

$T \equiv y = -x + a\frac{\sqrt{2}}{4}$

03. Find the derivative of the function described by the relation: $\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{a^2}$. 2 marks

Answer:

$\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{a^2} \Rightarrow x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{1/2} \Rightarrow \frac{1}{2}x^{\frac{1}{2}-1} + \frac{1}{2}y^{\frac{1}{2}-1}y' = 0 \Rightarrow y' = -\frac{x^{-\frac{1}{2}}}{y^{-\frac{1}{2}}} \Rightarrow y' = -\frac{\sqrt{y}}{\sqrt{x}}$

04. Solve the equations simultaneously:

$\begin{cases} 2^{\frac{1}{x}}2^{\frac{1}{y}} = 32 \\ 2^x 2^y = \sqrt[6]{32} \end{cases}$ 4 marks

Answer:

$\begin{cases} 2^{\frac{1}{x}}2^{\frac{1}{y}} = 32 \\ 2^x 2^y = \sqrt[6]{32} \end{cases}$

$\begin{cases} 2^{\frac{1}{x}+\frac{1}{y}} = 2^5 \\ 2^{x+y} = 2^{5/6} \end{cases}$

$\begin{cases} \frac{1}{x} + \frac{1}{y} = 5 \\ x + y = \frac{5}{6} \end{cases}$

$\begin{cases} \frac{x+y}{xy} = 5 \\ y = \frac{5}{6} - x \end{cases}$

$\begin{cases} \frac{5}{6} = 5x(\frac{5}{6} - x) \\ y = \frac{5}{6} - x \end{cases}$

$\begin{cases} \frac{5}{6} = \frac{25x}{6} - 5x^2 \\ y = \frac{5}{6} - x \end{cases}$

$\Rightarrow -6x^2 + 5x - 1 = 0 \Rightarrow 6x^2 - 5x + 1 = 0$

$x_1 = \frac{1}{2} \Rightarrow y_1 = \frac{1}{3}$

$x_2 = \frac{1}{3} \Rightarrow y_2 = \frac{1}{2}$

$S = \left\{\left(\frac{1}{2}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{1}{2}\right)\right\}$

05. Find the equations of the normal to the line whose equations is: 4x+6y-9=0

Answer:

Consider the equation of line $ax + bx + c = 0$

The normarsation factor is $\pm\frac{1}{\sqrt{a^2+b^2}}$

The equations of the normal are

$\frac{a}{\sqrt{a^2+b^2}}x + \frac{b}{\sqrt{a^2+b^2}}y + \frac{c}{\sqrt{a^2+b^2}} = 0$

$-\frac{a}{\sqrt{a^2+b^2}}x - \frac{b}{\sqrt{a^2+b^2}}y - \frac{c}{\sqrt{a^2+b^2}} = 0$

$a = 4, b = 6, c = -9, a^2 + b^2 - 52$

Then

$$D \equiv \frac{4}{\sqrt{52}}x + \frac{6}{\sqrt{52}}y - \frac{9}{\sqrt{52}} = 0$$

Or

$$D \equiv -\frac{4}{\sqrt{52}}x - \frac{6}{\sqrt{52}}y + \frac{9}{\sqrt{52}} = 0$$

06. Integrate: $\int_0^1 \frac{x}{(1+x^2)^2}dx$. 6 marks

Answer:

$\int_0^1 \frac{x}{(1+x^2)}dx$

Let $t = x^2$

$dt = 2xdx$

$xdx = \frac{dt}{2}$

if $x = 0 \Rightarrow t = 0$

$x = 1 \Rightarrow t = 1$

$I = \int_0^1 \frac{x}{(1+x^2)^2}dx = \frac{1}{2}\int_0^1 \frac{dt}{(1+t^2)^2}$

$\frac{1}{(1+t^2)^2} = \frac{1+t^2-t^2}{(1+t^2)^2} = \frac{1}{1+t^2} - \frac{t^2}{(1+t^2)^2}$

$\frac{1}{2}\int_0^1 \frac{dt}{(1+t^2)^2} = \frac{1}{2}\int_0^1 \frac{dt}{1+t^2} - \frac{1}{2}\int_0^1 \frac{t^2dt}{(1+t^2)^2}$

$I_1 = \frac{1}{2}\int_0^1 \frac{dt}{1+t^2} = \frac{1}{2}[\arctan t]_0^1 = \frac{1}{2}\left[\frac{\pi}{4} - 0\right] = \frac{\pi}{8}$

$I_2 = \frac{1}{2}\int_0^1 \frac{t^2dt}{(1+t^2)^2} =$

Let $u = t \Rightarrow du = dt$

$dv = \frac{tdt}{(1+t^2)^2} \Rightarrow v = \frac{1}{2(1+t^2)}$

$I_2 = \frac{1}{2}\int_0^1 t\frac{tdt}{(1+t^2)^2} = \frac{1}{2}\left\{\left[\frac{t}{2(1+t^2)}\right]_0^1 + \frac{1}{2}\int_0^1 \frac{dt}{1+t^2}\right\} = \frac{1}{2}\left\{\left[\frac{-1}{2(1+1^2)}\right]_0^1 + \frac{1}{2}\int_0^1 \frac{dt}{1+t^2}\right\} = \frac{1}{2}\left\{\left[\frac{-1}{4} + 0\right] + \frac{1}{2}\left[-\frac{1}{2}\arctan t\right]\right\} = \frac{1}{2}\left[-\frac{1}{4} + \frac{\pi}{8}\right] = -\frac{1}{8} + \frac{\pi}{16}$

$\frac{1}{2}\int_0^1 \frac{dt}{(1+t^2)^2} = \frac{\pi}{8} - (\frac{\pi}{16} - \frac{1}{8}) = \frac{\pi}{16} + \frac{1}{8}$

Second method

$\int_0^1 \frac{x}{(1+x^2)}dx$

Let $t = x^2$

$dt = 2xdx$

$xdx = \frac{dt}{2}$

if $x = 0 \Rightarrow t = 0$

$x = 1 \Rightarrow t = 1$

$I = \int_0^1 \frac{x}{(1+x^2)^2}dx = \frac{1}{2}\int_0^1 \frac{dt}{(1+t^2)^2}$

Let $t = \tan\theta$

$dt = \sec^2\theta d\theta$

if $t = 0 \Rightarrow \theta = 0$

$t = 1 \Rightarrow \theta = \frac{\pi}{4}$

$I=\frac{1}{2}\int_0^1\frac{dt}{(1+t^2)^2}=\frac{1}{2}\int_0^{\pi/4}\frac{sec^2\theta d\theta}{(1+tan^2\theta)^2}=\frac{1}{2}\int_0^{\pi/4}\frac{sec^2\theta d\theta}{(sec^2\theta)^2}=\frac{1}{2}\int_0^{\pi/4}\frac{1d\theta}{sec^2\theta}=\frac{1}{2}\int_0^{\pi/4}cos^2\theta d\theta$

$=\frac{1}{4}\int_0^{\pi/4}(1+cos\theta)d\theta=\frac{1}{4}\left[\theta+\frac{sin2\theta}{2}\right]_0^{\pi/4}=\frac{1}{4}\left(\frac{\pi}{4}-\frac{1}{2}\right)=\frac{\pi}{16}-\frac{1}{8}?$

07. About expanding the binomial $(3x^3-\frac{4}{x})^9$ find the coefficient of x^{-5}. 2 marks

Answer:

$(3x^3-\frac{4}{x})^9$ find the coefficient of x^{-5}

$C_k^9X^{9-K}Y^K=X^{-5}$

$\Rightarrow C_k^9(X^3)^{9-K}\left(\frac{1}{X}\right)=X^{-5}$

$X^{27-3K}X^{-K}=X^{-5}$

27-3K-K = -5

4K = 32

K = 8

$(X^3)^{9-K}=X^{-5}\Rightarrow 27-3K=-5\Leftrightarrow 32=3K$

$(X^3)(X^{-1})^8=X^{-5}$

$C_8^9(3X^3)(-4X^{-1})^8=C_8^9x3x4^8=1.769472$

08. Calculate the limits: a) $\lim_{x\to1^+}\left(\frac{1}{lnx}-\frac{1}{x-1}\right)$. 2 marks

b) $\lim_{x\to\infty}\left(\frac{n+2}{n+4}\right)^{4n}=e^{-2}$ 3 marks

Answer:

a) $\lim_{x\to1^+}\left(\frac{1}{lnx}-\frac{1}{x-1}\right)=\lim_{x\to1^+}\left(\frac{1}{0}-\frac{1}{0}\right)=\infty-\infty\ I.F$

Then $\lim_{x\to1^+}\frac{(x-1)-lnx}{lnx.(x-1)}=\lim_{x\to1^+}\frac{(1-1)-ln1}{lnx.(1-1)}=\frac{0}{0}\ I.F$

Then $\lim_{x\to1^+}\frac{((x-1)-lnx)'}{(lnx.(x-1))'}=\lim_{x\to1^+}\frac{(x-1)'-(lnx)'}{(lnx)'(x-1)'}=\lim_{x\to1^+}\frac{1-\frac{1}{x}}{\frac{x-1}{x}+lnx}=\frac{0}{0}\ I.F.$

Then $\lim_{x\to1^+}\frac{(1-\frac{1}{x})'}{(\frac{x-1}{x}+lnx)'}=\lim_{x\to1^+}\frac{\frac{1}{x^2}}{\frac{x-1}{x^2}+\frac{1}{x}}=\lim_{x\to1^+}\frac{\frac{1}{x^2}}{\frac{x^2-x+1}{x^2}}=\lim_{x\to1^+}\frac{1}{x^2-x+1}=1$

09. a) Determine the module and argument of $(1+i)^n$ and $(1-i)^n$, $n\in IN$. 2 marks

b) Deduce $(1+i)^n+(1-i)^n$ and $(1+i)^n-(1-i)^n$

Answer:

a) Let suppose this equation is equal to Z

$Z=(i+i)^n$ and $(1^0-i)^n=Z^*$

$|Z|=\sqrt{2}$ and $|Z^*|=\sqrt{2}$

$Cos\theta=\frac{\sqrt{2}}{4}\Leftrightarrow\theta=\frac{\pi}{4}$

$\sin\theta=\frac{\pi}{4}$

$\begin{cases}cos\theta=\frac{\pi}{4}\\ sin\ \theta=\frac{\pi}{4}\end{cases}$

b) $Z=\sqrt{2}\left(cos\frac{\pi}{4}+isin\frac{\pi}{4}\right)$

$Z^n=\left(\sqrt{2}(cos\frac{\pi}{4}+isin\frac{\pi}{4})\right)^n$

$Z = (\sqrt{2})^n (\cos n\frac{\pi}{4} + i\sin n\frac{\pi}{4})$

$Z^{*n} = \sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$

$Z^* = \left(\sqrt{2}(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4})\right)$

$Z^{*n} = (\sqrt{2})^n (\cos n\frac{\pi}{4} - i\sin n\frac{\pi}{4})^n$

$Z^* = (\sqrt{2})^n (\cos n\frac{\pi}{4} - i\sin n\frac{\pi}{4})$

$Z+Z^* = 2(\sqrt{2})^n\, 2\cos n\frac{\pi}{4} = 2^2.2^{n/2}\cos n\frac{\pi}{4} = 2^{\frac{2n+1}{n}}\cos n\frac{\pi}{4}$

$Z - Z^* = 2(\sqrt{2})^n (\cos n\frac{\pi}{4} + i\sin n\frac{\pi}{4}) - 2(\sqrt{2})^n (\cos n\frac{\pi}{4} - i\sin n\frac{\pi}{4})$

$= 2^{\frac{2n+1}{n}} i\sin n\frac{\pi}{4}$

10. Determine the parametric equations and the Cartesian equations of the line D_2 which passes through point (-3; 5; 7) and is parallel to line D_1, of the equations

$\begin{cases} x + y = 1 \\ 4x - z = 0 \end{cases}$ 4 marks

Answer:

$D \equiv \begin{cases} x + y = 1 \\ 4x - z = 0 \end{cases}$

$\Rightarrow \begin{cases} x = 1 - y \\ x = \frac{z}{4} \end{cases}$

Or

$\frac{x}{1} = \frac{y-1}{-1} = \frac{z}{4}$

Then D passes through the point (0, 1, 0) and has director vector (1, -1, 4) like D_2 is parallel to D since they have the same director vector.

The parametric equations of D_2 are:

$D_2 \equiv \begin{cases} -3 + r \\ 5 - 5 \\ 7 + 4r \end{cases}$

Cartesian equations of D_2

$D \equiv \frac{x+3}{1} = \frac{y-5}{-1} = \frac{z-7}{4}$

11. For which values of a and b are the points M(1; 2; 4), N(a; 4; 8) and T(-3; -6; b) collinear? 3 marks

Answer:

The points M(1, 2, 4), N(a, 4, 8) and T(-3, -6, b) are collinear if the vectors $\overrightarrow{MN}$ *and* $\overrightarrow{MT}$ are parallel

e.g: $\overrightarrow{MN} = \lambda\, \overrightarrow{MT}$

(a-1, 2, 4) = $\lambda(-4, -8, b-4)$

$\begin{cases} a - 1 = -4\lambda \\ 2 = -8\lambda \\ 4 = \lambda(b - 4) \end{cases}$

$\Rightarrow \lambda = \frac{-1}{4}$

$\Rightarrow$ a = 2 and b = -12

Other method

$$\begin{vmatrix} 1 & a & -3 \\ 2 & 4 & -6 \\ 4 & 8 & b \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} 1 & a & -3 \\ 0 & 4-2a & 0 \\ 4 & 8 & b \end{vmatrix} = 0$$

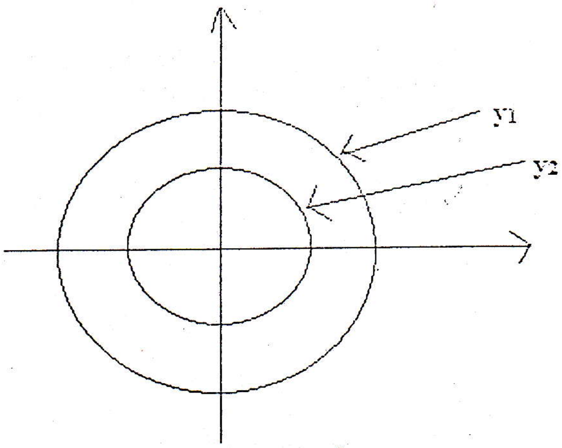
$\Leftrightarrow$ (4-2a)(b+12) = 0 $\Leftrightarrow$ a = 2, b = -12

12. Consider two circles: $C_1: x^2 + y^2 = 4$
$C_2: x^2 + y^2 = 9$

a) Find the surface area bounded by the two circles. 4 marks
b) If this surface area is revolved about OX axis, find the volume of revolution obtained. 4 marks

Answer:

a) $y_1 = \sqrt{4 - x^2}$ and $y_2 = \sqrt{9 - x^2}$



$S = 2\left[\int_{-3}^{3} y_2 dx - \int_{-3}^{3} y_1 dx\right]$

$I_1 = \int_{-3}^{3} y_2 dx = \int_{-3}^{3} \sqrt{9 - x^2} dx$

Let x = 3cost $\Leftrightarrow$ dx = -3sintdt

If x = -3 $\Rightarrow$ t = π

x = 3 $\Rightarrow$ t = 0

$I_1 = -3\int_{\pi}^{0} \sqrt{9 - 9cos^2t}\, sintdt = 9\int_0^{\pi} sin^2tdt = 9\int_0^{\pi} \frac{(1-cos2t)dt}{2} dt = \frac{9}{2}\left[\frac{t-sin2t}{2}\right]_0^{\pi} = \frac{9\pi}{2}$

$I_2 = \int_{-2}^{2} y_1 dx = \int_{-2}^{2} \sqrt{4 - x^2} dx$

Let x = 2cost $\Leftrightarrow$ dx = -2sintdt

If x = -2 $\Rightarrow$ t = π

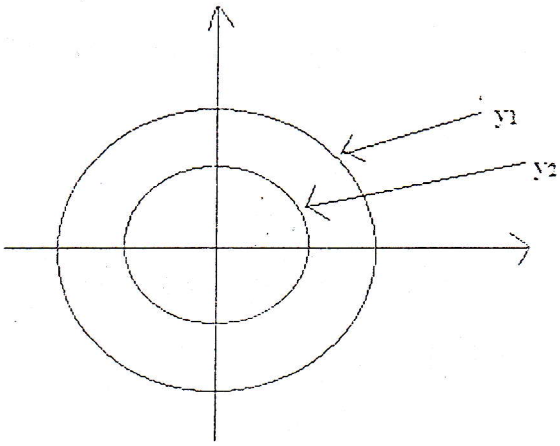
x = 2 $\Rightarrow$ t = 0

$I_1 = -2\int_{\pi}^{0} \sqrt{4 - 4cos^2t}\, sintdt = 4\int_0^{\pi} sin^2tdt = 4\int_0^{\pi} \frac{(1-cos2t)dt}{2} dt = \frac{4}{2}\left[\frac{t-sin2t}{2}\right]_0^{\pi} = \frac{2\pi}{2} = 2\pi$

$S = I_1 - I_2 = \frac{9\pi}{2} - 2\pi = 5\pi$

Another method:

$S_1 \equiv C_1 = x^2+y^2 = 4$ $\qquad$ $S_2 \equiv C_2 = x^2+y^2 = 9$



$S = S_2 - S_1$

$S_1 = \pi r^2 = 4\pi$ and $S_2 = \pi r^2 = 9\pi$

$S = 9\pi - 4\pi = 5\pi$

b) $V = V_2 - V_1 = \pi\left[\int_{-3}^{3} y_2^2 dx - \int_{-2}^{2} y_1^2 dx\right]$

$V_1 = \pi \int_{-2}^{2} y_1^2 dx = \pi \int_{-2}^{2} (4 - x^2) dx = \frac{32\pi}{3}$

$V_2 = \pi \int_{-3}^{3} y_2^2 dx = \pi \int_{-3}^{3} (9 - x^2) dx = \frac{108\pi}{3}$

$V = \frac{108\pi}{3} - \frac{32\pi}{3} = \frac{76\pi}{3}$ UV

13. Let $\sigma_{xy} = 0.84$; $\sigma_x = 1.25$; $\sigma_y = 1.65$ be coefficient of linear correlation and the standard deviations of a group of 10 students whose weights are (X kg) and height are (Y cm).
    a) Find the equation of each of the lines of regression if $\overline{x} = 67$ kg and $\overline{y} = 160$
    b) Estimate the weight of a student who is 172 cm tall and the height of a student who weighs 70 kg. **1 mark**

Answer:

$\sigma_{xy} = 0.84$ $\delta_x = 1.25$; $\delta_y = 1.65$

$\sigma_{xy} = \frac{cov(x,y)}{\delta_x \delta_y}$

Where $cov(x, y) = \sigma_{xy} . \delta_x . \delta_x$

$= (0.84)(1.25)(1.65) = 1.7325$

$\overline{x} = 67^o$ $\overline{y} = 160^0$

a) $y = \overline{y} + \frac{cov(x,y)}{\delta_x{}^2}(x - \overline{x}) = 160 + \frac{1.7325}{1.25^2}(x - 67) = 85.4104 + 1.1088x$

Equation of regression line of x in y

$x = \overline{x} + \frac{cov(x,y)}{\delta_y{}^2}(y - \overline{y}) = 67 + \frac{1.7325}{1.65^2}(y - 160) = -34.8181 + 0.636x$

b) If y = 172m then x =?

$x = -34.8181 + 0.636(172) = 74.6$ kg

If x = 70

$y = 85.4104 + 1.1088(70) = 163.32$cm

14. Two students want to solve the same problem without consulting each other. The probability of the first student to find the correct solution is 0.8 where the probability of

the second student to find the correct solution is 0.2. What is the probability of solving without consulting each other....5 marks

Answer:

Let E_1 the event of the first student

E_2 the event of the second student

E_1 and E_2 are not consulting $\Rightarrow$ E_1 and E_2 are independent

$R = E_1 \cup E_2$

$P(R) = P(E_1 \cup E_2) = P(E_1)+P(E_2)-P(E_1 \cap E_2) = P(E_1)+P(E_2)-P(E_1).P(E_2) = 0.8 + 0.2 - 0.8 \times 0.2 = 0.84$

15. Find the paramagnetic equations and Cartesian equations of the straight line D which passes through point (3, -2, 5) and in direction vector $\begin{pmatrix}3\\4\\6\end{pmatrix}$. 2 marks

Answer:

$\vec{v} = (3, 4, 6)$

$a = (3, -2, 5)$

Parametric equations of line

$x = 3+3\lambda$

$y = -2+4\lambda$

$z = 5+6\pi$

Cartesian equation

$\frac{x-3}{3} = \frac{y+2}{4} = \frac{z-5}{6}$

SECTION B: Choose three questions in this section (45 marks each)

16. Let $I_n = \int_2^n \frac{x\ln x}{(x^2-1)^2}dx$, n>2

a) Determine a, b, c such that $\frac{1}{x(x^2-1)} = \frac{a}{x} + \frac{b}{x-1} + \frac{c}{x+1}$ 3 marks

b) Calculate $\int_2^n \frac{dx}{x(x^2-1)^2}$. 5 marks

c) With aid of integration by parts calculate I_n. 2.5 marks

d) Calculate $\lim_{n\to\infty} I_n$. 4 marks

Answer:

a) $\frac{1}{x(x^2-1)} = \frac{a}{x} + \frac{b}{x-1} + \frac{c}{x+1}$

$= \frac{a(x^2-1)+bx(x+1)+cx(x-1)}{x(x^2-1)}$

$\begin{cases} a+b+c=0 \\ b-c=0 \\ -a=1 \end{cases}$

$\begin{cases} a=-1 \\ b=c \\ b+c=1 \end{cases}$

$\begin{cases} a=-1 \\ b=\frac{1}{2} \\ c=\frac{1}{2} \end{cases}$

b) $\frac{1}{x(x^2-1)} = -\frac{1}{x} + \frac{1}{2(x-1)} + \frac{1}{2(x+1)}$

$\int_2^n \frac{dx}{x(x^2-1)^2} = -\int_2^n \frac{dx}{x} + \frac{1}{2}\int_2^n \frac{dx}{x-1} + \frac{1}{2}\int_2^n \frac{dx}{x+1}$

$= -[lnx]_2^n + \frac{1}{2}[ln(x-1)]_2^n + [ln(x+1)]_2^n =$

$= \ln 2 - \ln n + \frac{1}{2}ln(n-1) + \frac{1}{2}ln(n+1) - ln\frac{ln3}{2} = \frac{1}{2}ln\frac{4}{3} + ln\frac{\sqrt{n^2-1}}{n}$

c) $I_n = \int_2^n \frac{xlnx}{(x^2-1)^2}dx$

Let $u = \ln x \Rightarrow du = \frac{dx}{x}$

$dv = \frac{xlnx}{(x^2-1)^2} \Rightarrow v = -\frac{1}{2(x^2-1)}$

$I_n = -\left[\frac{lnx}{2(x^2-1)}\right]_2^n + \frac{1}{2}\int_2^n \frac{dx}{x(x^2-1)^2} = -\frac{lnn}{2(n^2-1)} + \frac{ln2}{6} + \frac{1}{2}(b)$

$b = \int_2^n \frac{dx}{x(x^2-1)^2} = \frac{1}{2}ln\frac{4}{3} + ln\frac{\sqrt{n^2-1}}{n}$

$I_n = -\left[\frac{lnx}{2(x^2-1)}\right]_2^n + \frac{1}{2}\int_2^n \frac{dx}{x(x^2-1)^2} = -\frac{lnn}{2(n^2-1)} + \frac{ln2}{6} + \frac{1}{2}\left(\frac{1}{2}ln\frac{4}{3} + ln\frac{\sqrt{n^2-1}}{n}\right)$

d) $\lim_{n\to\infty} I_n = \left(-\frac{lnn}{2(n^2-1)} + \frac{ln2}{6} + \frac{1}{2}\left(\frac{1}{2}ln\frac{4}{3} + ln\frac{\sqrt{n^2-1}}{n}\right)\right) =$

$\lim_{n\to\infty} \frac{lnn}{2(n^2-1)} = \lim_{n\to\infty} \frac{\frac{1}{n}}{4n} = \lim_{n\to\infty} \frac{1}{4n^2} = 0$

And $\lim_{n\to\infty} ln\frac{\sqrt{n^2-1}}{n} = \lim_{n\to\infty} ln\frac{n\sqrt{1-\frac{1}{n^2}}}{n} = \ln 1 = 0$

Hence $\lim_{n\to\infty} I_n = \frac{ln2}{6} + \frac{1}{4}\frac{ln4}{3} = 0.18744$

17. Given the polar equation of the curve $\vartheta = \frac{9p}{4-5cos\theta}$.

a) Determine the concentricity and the nature of the curve. 4 marks
b) Find the distance from focal point (P= pole) to the derictrice near the pole. 2 marks
c) If $P = \frac{b^2}{a}$, transform the equation to Cartesian coordinates. 4 marks
d) Determine the equations of asymptotes and equations of derectrices in short form. 3 marks
e) Construct the curve. 2 marks

Answer:

a) $\vartheta = \frac{9p}{4-5cos\theta}$

$\vartheta = \frac{\frac{9}{4}}{4-\frac{5}{4}cos\theta} \Rightarrow e = \frac{5}{4} > 1$

The curve is hyperbola

b) $pxe = \frac{9}{4}$

$p \cdot \frac{5}{4} = \frac{9}{4}$

The direct rice is founded at $\frac{9}{4}$ unit at the left of pole

c) $e^2-1 = \frac{b^2}{a^2}$ *for hyperbola*

$\frac{b^2}{a^2} = \frac{25}{16} - 1 = \frac{9}{16}$

Or $\frac{b^2}{a^2} = p = \frac{9}{5}$

Then

$\begin{cases} \frac{b^2}{a^2} = \frac{9}{5} \\ \frac{b^2}{a^2} = \frac{9}{16} \end{cases}$

$\Rightarrow a = \frac{a^2}{b^2} = \frac{\frac{b^2}{a^2}}{\frac{b^2}{a^2}} = \frac{\frac{9}{5}}{\frac{9}{16}} = \frac{16}{5}$

$b^2 = ap = \frac{16}{5} x \frac{9}{16} = \frac{144}{25}$

$b = \frac{12}{5}$

d) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\frac{x^2}{\frac{256}{25}} - \frac{y^2}{\frac{144}{25}} = 1$

Asymptote equations are

$y = \pm\frac{b}{a}x$

$y = \pm\frac{\frac{12}{5}}{\frac{16}{5}}x$

$y = \pm\frac{12}{16}x = \pm\frac{3}{4}x$

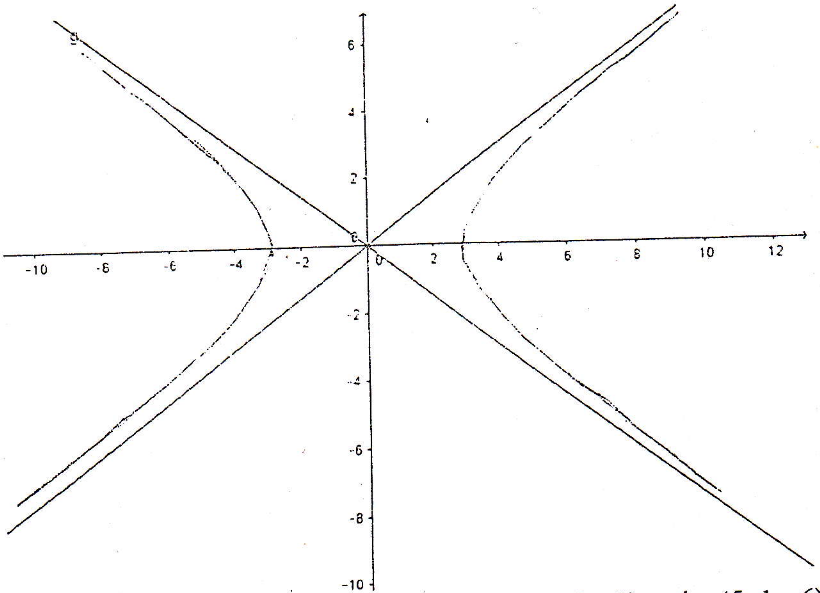
$Y = \pm\frac{3}{4}x$

$y_1 = +\frac{3}{4}x$ *and* $y_2 = -\frac{3}{4}x$

Derectrices equations are

$D \equiv x = \pm\frac{a}{e} = \frac{\pm\frac{16}{5}}{\pm\frac{5}{4}} = \pm\frac{64}{25}$

e)



18. Given the points a (1, 1, 1), b (2, 3, 4), c (3, -1, 4), p (3, 0, -3) and q (5, 1, -6). The coordinates of point m which belongs to the plane abc and on line pq are to be determined in as many ways as possible.
    a) Write the parametric equations of plane abc and the line pq. Deduce the value of parametres of point m and the coordinates of m. **2 marks**
    b) Write Cartesian equations of plane abc and of line pq. Deduce the coordinates of m. **13 marks**

Answer:

a) Equation of plane — **Parametric equation**

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\3\end{pmatrix} + S\begin{pmatrix}2\\-2\\3\end{pmatrix} \qquad \begin{cases}x-1=\lambda+2S\\ y-1=2\lambda-2S\\ z-1=3\lambda+3S\end{cases}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}\lambda\\2\lambda\\3\lambda\end{pmatrix} + \begin{pmatrix}2S\\-2S\\3S\end{pmatrix} \qquad \begin{cases}x=1+\lambda+2S\\ y=1+2\lambda-2S\\ z=1+3\lambda+3S\end{cases}$$

$$\Rightarrow \begin{cases}x-1=\lambda+2S\\ y-1=2\lambda-2S\\ z-1=3\lambda+3S\end{cases} \Rightarrow \begin{vmatrix}x-1 & y-1 & z-1\\ 1 & 2 & 3\\ 2 & -2 & 3\end{vmatrix}$$

$$(x-1)\begin{vmatrix}2 & 3\\ -2 & 3\end{vmatrix} - y - 1\begin{vmatrix}1 & 3\\ 2 & 3\end{vmatrix} + z - 1\begin{vmatrix}1 & 2\\ 2 & -2\end{vmatrix}$$

(x-1)12 –(y-1)(-3) + (z-1)(-6) = 12x-12-(-3y+3) +(-6z+6) = 12x-12+3y-3-6z-6 = 0

⇒ **12x+3y-6z-9 = 0 Equation of plane**

Parametric equation of line

The point pq = p(3, 0, -3) q(5, 1, -6)

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}3\\0\end{pmatrix} + t\begin{pmatrix}2\\1\end{pmatrix} \Rightarrow \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}3\\0\end{pmatrix} + \begin{pmatrix}2t\\t\end{pmatrix}$$

$$\begin{cases} x = 3 + 2t \\ y = t \\ z = -3 + 9t \end{cases} \Leftrightarrow \begin{cases} x - 3 = 2t \\ y = t \\ z + 3 = 9t \end{cases}$$

Cartesian equation

$12x+3y-6z-9 = 0$

Then $12(3+2t)+3t-6(-3+9t)-9 = 0 \Rightarrow 36+24t+3t+18-54t-9 = 0 \Rightarrow -27t = -45$

$t = \frac{45}{27} = 1.66 \cong 1.7 \cong 2$

$$\begin{cases} x = 3 + 2.2 \\ y = 2 \\ z = -3 + 9.2 \end{cases} \Leftrightarrow \begin{cases} x = 7 \\ y = 2 \\ z = 15 \end{cases}$$

The point m = (7, 2, 15)

19. In a bag there are n red marbles ($n \geq 2$) and one white marble.

a) Calculate the probability $p_1(n)$ to pick one red marble if only one marble has to be picked randomly. Show that $p_1(n)$ is an ascending sequence and calculate $\lim_{x \to \infty} P_1(n)$. 5 marks

b) Calculate the probability $p_2(n)$ of picking 2 red marbles if two marbles have to be picked randomly at once from the bag. Show that $p_2(n)$ is an ascending sequence and calculate $\lim_{x \to \infty} P_2(n)$. 5 marks

c) Calculate the difference $P_1(n) - P_2(n)$. 2 marks

d) Determine the set of values of n for which $P_1(n) - P_2(n) < \frac{1}{8}$. 3 marks

Answer:

n red marbles ($n \geq 2$) and one white marble

Let n+1 marbles

They picked randomly one marble

a) $\neq \Omega_1 = \binom{1}{n+1}$

$P_1(n) = \frac{\binom{1}{n}}{\binom{1}{n+1}} = \frac{n}{n+1}$

Let $u_n = P_1(n) = \frac{n}{n+1}$

Show that u_n is an ascending sequence means $u_{n+1} - u_n > 0$

$u_{n+1} = \frac{n+1}{n+2}$

$u_{n+1} - u_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{(n+1)^2 - n(n+2)}{(n+2)(n+1)} = \frac{n^2+2n\mp 1-n^2-2n}{(n+2)(n+1)} = \frac{1}{(n+2)(n+1)} > 0$

Hence u_n is increasing

$\lim_{x \to \infty} P_1(n) = \lim_{x \to \infty} \frac{n}{n+1} = 1$

b) $\neq \Omega_1 = \binom{2}{n+1}$

$P_2(n) = \frac{\binom{2}{n}}{\binom{1}{n+1}} = \frac{n-1}{n+1}$

Let $V_n = P_2(n) = \frac{n-1}{n+1}$

Show that V_n is an ascending sequence means $V_{n+1} - V_n > 0$

$V_{n+1} = \frac{n}{n+2}$

$V_{n+1}-V_n = \frac{n}{n+2} - \frac{n-1}{n+1} = \frac{(n+1)n-(n-1)(n+2)}{(n+1)(n+2)} = \frac{n^2+n-n^2-n+2}{(n+2)(n+1)} = \frac{2}{(n+2)(n+1)} > 0$

Hence $V_n = P_2(n)$ is increasing

$\lim_{x\to\infty} P_2(n) = \lim_{x\to\infty} \frac{n-1}{n+1} = 1$

c) $P_1(n)-P_2(n) = \frac{n}{n+1} - \frac{n-1}{n+1} = \frac{1}{n+1}$

d) $P_1(n)-P_2(n) < \frac{1}{8} \Leftrightarrow \frac{1}{n+1} < \frac{1}{8} \Leftrightarrow 8 < n+1 \Leftrightarrow n = 7$

20. ...

Answer:

a)

$f(e_1) = 3e_1 - e_2$

$f(e_2) = e_1 3e_2$

Mf is the matrix of f

$Mf = (f(e_1).f(e_2)) = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}$

$\vec{x} = x_1\vec{e}_1 - x_2\vec{e}_2$

$f(\vec{x}) = Mf.x = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (3x_1 + x_2 - x_1 + x_2)$

$Then\ f(\vec{x}) = (3x_1 + x_2)\vec{e}_1 + (-x_1+3x_2)\vec{e}_2$

$The\ same f(\vec{y}) = (3y_1 + y_2)\vec{e}_1 + (-y_1+3y_2)\vec{e}_2$

$f(\vec{x}).f(\vec{y}) = (3x_1 + x_2 - x_1 + 3x_2)\begin{pmatrix} 3y_1 + y_2 \\ -y_1 + 3y_2 \end{pmatrix} = (3x_1 + x_2)(3y_1 + y_2)(-x_1 + 3x_2)(-y_1 + 3y_2) = 9x_1y_1 + 3x_1y_2 + 3y_1x_2 + 2x_2y_2 + x_1y_1 - 3x_1y_2 + 3x_2y_1 + 9x_2y_2$

$= 10x_1y_1 + 10x_2y_2 = 10(x_1y_1 + x_2y_2)$

$= 10(x_1 + x_2)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 10\vec{x}.\vec{y}$

b) $a' = f(\vec{a}) = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} -1 \\ -1 \end{pmatrix} = (3+1, -1+3) = (4, 2)$

$b' = f(\vec{b}) = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} -1 \\ -3 \end{pmatrix} = (-3-3, 1-9) = (-6, -8)$

c) $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2$

$a\vec{x} = (1 - x_1)\vec{e}_1 + (1 - x_2)\vec{e}_2$

$b\vec{x} = (-1 - x_1)\vec{e}_1 + (-3 - x_2)\vec{e}_2$

$\vec{x}a.\vec{x}b = (-1 - x_1, 1 - x_2)\begin{pmatrix} -1 - x_1 \\ -3 - x_2 \end{pmatrix} = (1 - x_1)(1 - x_1) + (1 - x_2)(-3 - x_2)$

$= (-1 - x_1 + x_1^2 + (-3 - x_2 + 3x_2 + x_2^2)$

$= x_1^2 + +x_2^2 + 2x_2 + 4$

$\vec{x}a.\vec{x}b = 0 \Leftrightarrow x_1^2 + +x_2^2 + 2x_2 + 4 = 0 \Leftrightarrow x_1^2 + (x_2 + 1)^2 - 1 - 4 = x_1^2 + (x_2 + 1)^2 = 5$

is the circle of the centre (0, -1) and the radius $\sqrt{5}$

ADVENCED LEVEL MATHEMATICS NATIONAL EXAMINATION PAPER 2008
(BIOLOGY-CHEMISTRY and COMPUTER SCIENCE)

SECTION A: Attempt all questions. (55 marks)

01. Let ⊥ be a binary operation defined on the ring of integers (Z) by a ⊥ b = a+b-1

a) Calculate (-10) ⊥ (-2); 2 ⊥ (-7).
b) Is the given binary operation commutative? Associative?
c) Determine whether there is an identity element in Z for ⊥.
d) If there is an identity element, which elements have inverse?
e) Conclude on (Z, ⊥). 5 marks

Answer:

Answer:

a) (-10) ⊥ (-2) = (-10)+(-2) -1 = -13
2 ⊥ (-7) = 2+(-7) -1 = -6
b) a ⊥ b = a+b-1 = b+a-1 = b⊥ $a \Rightarrow \perp$ $is\ commutative$
c) $\perp$ $is\ associative\ if\ \forall\ a, b, c \in Z\ such\ that\ (a \perp b) \perp c = a \perp (b \perp c)$
$(a \perp b) \perp c = (a + b - 1) \perp c = (a + b - 1) + c - 1$
= a+b+c-1-1
= a+(b+c-1)-1
= a+b(b⊥ c) = -1
= a⊥ $(b + c) \Rightarrow$ $is\ associative$

d) The element e∈Z is said neutral with the law ⊥ $if \forall\ a \in Z\ such\ that\ a \perp e = a =$
$a = e \perp a$
As ⊥ $is\ commutative, it\ is\ enough\ that\ a \perp e = a\ therefore\ a \perp e = 1 \Rightarrow$
$a \perp a + e - 1 = a \Leftrightarrow e = 1$
Thus the neutral element for ⊥ is 1
Let find if a admits an inverse a' such that a⊥ $a' = 1$
It is enough again that a⊥ $a' = 1 \Leftrightarrow a + a' - 1 = 1 \Leftrightarrow a' = -a + 2$
The inverse (or the symmetric) of a is thus –a+2 for every integer a of Z

e) (Z, ⊥) $is\ a\ commutative\ group\ (or\ abelian)$

02. a) Verify by differentiation that $\int xsinx\ dx = sinx - xcosx + C$ with C an arbitrary real.

b) Find the volume of the solid obtained by rotating around the y-axis the area under y=sinx from x = 0 to x = π. 3 marks

Answer:

a) (sinx – xcosx+c)' = cos x – cosx + xsinx = xsinx, the result is true
b)